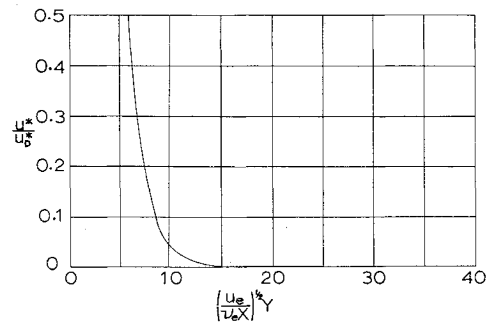

Fig. 4 Velocity profile in the physical coordinate ($\omega = 1.0$).

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Creep and Relaxation

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In performing analyses in viscoelasticity, it is often necessary to convert a known relaxation modulus $E(t)$ to its associated creep compliance $J(t)$ and vice versa. Hopkins and Hamming[1] have presented a step-by-step numerical integration method for obtaining one material function when the other is known. In their computations, the interrelationship between $E(t)$ and $J(t)$ is taken in the form

$$\int_{-\infty}^{t} E(t - t')\, J(t')\, dt' = t\, H(t) \tag{1}$$

where $H(t)$ is the Heaviside unit step function. The role of $E(t)$ and $J(t)$ may be interchanged in (1). Utilizing this form of the interconversion, it is necessary to define the integral of one of the functions in order to perform the numerical integration. If the interrelationship between $E(t)$ and $J(t)$ is taken in the alternative form

$$\int_{-\infty}^{t} E(t - t')\, \frac{dJ(t')}{dt'}\, dt' = H(t) \tag{2}$$

numerical integration may be performed without introducing the integral of one of the functions. To this end, the discontinuity term in (2) is removed, and (2) is integrated by parts, yielding

$$J(t)\, E(0) - \int_0^t J(t')\, \frac{dE(t - t')}{dt'}\, dt' = 1 \tag{3}$$

Using the step-by-step integration procedure proposed by Lee and Rogers,[2] Eq. (3) may be solved for $J(t_n)$, where t_n corresponds to the present time t. As before, if it is desired to determine $E(t)$ from a known $J(t)$, the role of E and J may be interchanged in (3). As stated in Ref. 2, an advantage is gained by using this form of the material property interrelationship since the form of the time step becomes arbitrary and may be logarithmic, constant, etc.

If secondary creep exists in $J(t)$, the convergence requirements for numerical integration may break down, and the solution may become unstable. To circumvent this difficulty, the secondary creep is removed by defining

$$J(t) = S(t) + Rt \tag{4}$$

where $J(0) = S(0)$ and $R = \dot{J}(\infty)$. A ($\dot{}$) denotes differentiation with respect to time. Thus, in (4), $S(t)$ is now a bounded function for all time. Substitution of (4) into (3) yields, after simplification,

$$S(t)\, E(0) - \int_0^t S(t')\, \frac{dE(t - t')}{dt'}\, dt' = 1 - R \int_0^t E(t')\, dt' \tag{5}$$

The constant R, i.e., $\dot{J}(\infty)$, may be evaluated by integrating the left-hand side of (5) by parts and taking the limit as t goes to infinity. Upon noting that, for secondary creep to be present, $E(\infty)$ is zero, and by the definition of $S(t)$, $S(\infty)$ is zero, the integral term may be shown to vanish, and

$$R = 1 \Big/ \int_0^\infty E(t)\, dt \tag{6}$$

Care must be exercised in using (6) as it is valid only if the secondary creep is present; otherwise, R is identically zero.

Step-by-step numerical integration of (5) by the scheme proposed in Ref. 2 yields

$$\tfrac{1}{2}\{E(0) + E(t_n - t_{n-1})\}\, S(t_n) = 1 - R\, F(t_n) + \tfrac{1}{2} J(0)\, \{E(t_n - t_1) - E(t_n - t_{i-1})\} + \frac{1}{2}\sum_{i=1}^{n-1} J(t_i)\, \{E(t_n - t_{i+1}) - E(t_n - t_{i-1})\} \qquad n > 1 \tag{7}$$

where

$$F(t_n) = \int_0^{t_n} E(t')\, dt' \tag{8}$$

For $t = 0$ and $t = t_1$,

$$S(0) = J(0) = 1/E(0)$$

and

$$S(t_1) = \frac{S(0)\, [3\, E(0) - E(t_1)] - 2\, R\, F(t_1)}{E(0) + E(t_1)} \tag{9}$$

Finally, the creep compliance is computed from (4) as follows:

$$J(t_n) = S(t_n) + Rt_n \tag{10}$$

If no secondary creep exists, all terms multiplied by R vanish, and $S(t_n)$ is identically equal to $J(t_n)$.

As an example, the creep compliance is calculated from (7) for a material whose relaxation function is given by

$$E(t) = E_0 \left\{ \frac{1}{6} \int_t^{te^6} \frac{e^{-u}}{u}\, du \right\} \tag{11}$$

This relaxation function results from a problem defined in Refs. 3 and 4, and in Sec. 12 of Ref. 5 The numerical inversion is accomplished by direct inversion without removing the secondary creep and by the modified procedure presented herein. Constant logarithmic steps of 0.05 were used for both inversions. The results are shown in Fig. 1. The results are compared to the exact solution obtained from Ref. 4. As is seen from Fig. 1, direct numerical inversion yields results that are considerably in error from the exact solution, whereas the modified solution obtained by first removing the secondary creep agrees very well when compared to the exact solution (for the scale used in Fig. 1, no difference is visible).

This note has presented a numerical method for determining the creep compliance (when secondary creep is present) from its associated relaxation modulus. The numerical example presented indicates the error that can result from a direct inversion without first removing the secondary creep.

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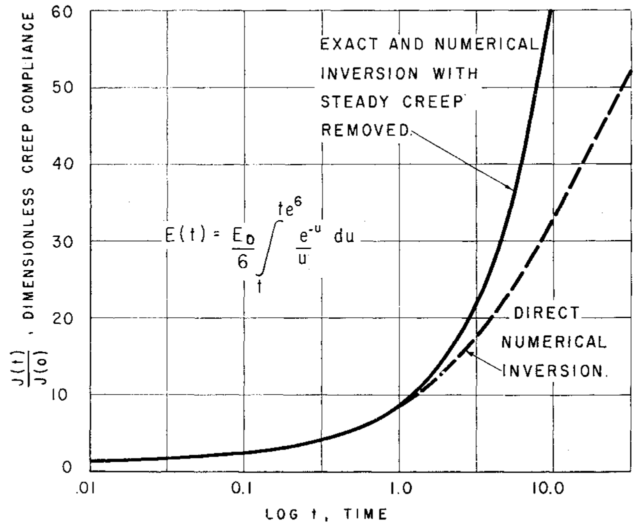

Fig. 1 Creep compliance with secondary creep.

The method herein presented may be extended and applied to any Volterra integral equation in which the form of the long-time response of the unknown function is known.

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Hypersonic Viscous Flow Near a Sharp Leading Edge

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Nomenclature

M = u_∞/a_∞ = freestream Mach number
Re_x = $\rho_\infty u_\infty x/\mu_\infty$ = Reynolds number
ϵ = $(\gamma - 1)/(\gamma + 1)$
λ = mean free path

Subscripts

∞ = conditions in freestream
s = conditions immediately behind shock
w = conditions at wall
0 = stagnation conditions

The hypersonic viscous flow near the sharp leading edge of a flat plate has been the subject of many studies in recent years. Several investigators[1, 2] have reported measurements of surface pressure and heat transfer which begin to deviate markedly from the values predicted by viscous strong interaction theory as the leading edge is approached. Pressure and heat-transfer plateaus have been observed with levels well below those corresponding to strong interaction theory. The purpose of this note is to suggest that, when $(\epsilon l\ ^2)^{1/2} \gg 1$, as in many cases of interest, these departures are not attributable to slip phenomena (except very near the leading edge) but rather to a viscous flow of more complicated nature than that of the strong interaction region. The mass flow is expected to divide nearly equally between a thin inviscid layer near the shock and a thick viscous layer below, across which the pressure is not constant and where the boundary layer approximation is not valid.

For a hypersonic flow, Hayes and Probstein (Chap. 10 of Ref. 3) use a mass conservation argument to point out that, upstream of the strong interaction region, the shock wave should become straight. Good evidence for this argument appears in the experimental work of Chuan and Waiter.[4] The shock inclination should be of the order $(\rho_\infty/\rho_s)^{1/2}$ or, for a perfect gas, $y_s/x \sim \epsilon^{1/2}$ (see Fig. 1). If a strong interaction region exists downstream, the shock wave in this region must have a large normal Mach number. The square of the normal Mach number in this straight shock or viscous layer region (as it will be termed in this note) is of the order $(My_s/x)^2 \sim \epsilon M^2$. A condition for the existence of this region is then $\epsilon M^2 \gg 1$. The pressure in the region should be reasonably uniform (since the shock is nearly straight) and of the order $p_\infty \epsilon M^2$. The upstream boundary of this region should be where noncontinuum phenomena begin to dominate the flow, whereas the downstream limitation of the viscous layer region should occur where strong interaction theory becomes valid. The pressure level for strong interaction flow is of the order $p_\infty \epsilon M^3/Re_x^{1/2}$, whereas that for the viscous layer flow is of the order $p_\infty \epsilon M^2$. Since the strong interaction pressure should always be less than the viscous layer pressure, the boundary between the two regions should occur for $M/Re_x^{1/2} \lesssim O(1)$. It is shown below that the upstream limit for the viscous layer region should occur where $M^2/Re_x \lesssim O(\epsilon M^2)^{1/2}$. A value for $M/Re_x^{1/2}$ of order unity ought then to characterize the viscous layer region.

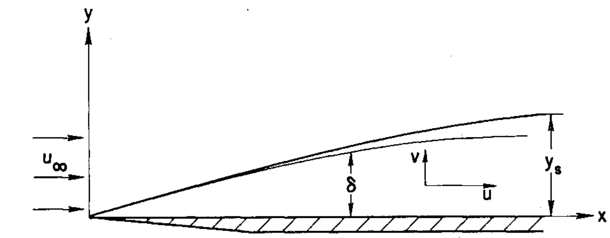

Fig. 1 Coordinate system for leading edge flow.

From schlieren photographs[1] the viscous layer region appears to have a viscous zone that nearly fills the entire shock layer. In this viscous zone, severe dissipation heats the fluid to nearly stagnation temperatures. Then the average temperature there would be of the order $T_\infty \epsilon M^2$. Since the pressure level is of the order $p_\infty \epsilon M^2$, the density will be of order ρ_∞ except very close to a cold wall.

Assuming for the moment that a thin shock adequately described by the Rankine-Hugoniot relations exists, there will be a thin zone of relatively cool inviscid flow between the shock and the viscous zone where the state properties are essentially those found immediately behind the shock. Thus, the density here will be of the order ρ_∞/ϵ, whereas the horizontal velocity will be of the order u_∞.

Over-all mass conservation requires the following relation:

$$\rho_\infty u_\infty y_s = \int_0^\delta \rho u dy + \int_\delta^{y_s} \rho u dy \qquad (1)$$

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